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Algebras of Subnormal Operators*

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The paper deals with the following: (I) If S is a subnormal operator on $\mathscr{H}$, then $\mathscr{A}(S) = \mathscr{W}(S) =$ Alg Lat S. (II) If $L \in (\mathscr{A}(S), \sigma\text{-wot})^*$, then there exist vectors a and b in $\mathscr{H}$ such that $L(T) = \langle T a, b\rangle$ for every T in $\mathscr{A}(S)$. (III) In addition to I the map $i(T) = T$ is a homeomorphism from $(\mathscr{A}(S), \sigma\text{-wot})$ onto $(\mathscr{W}(S), \text{wot})$. (IV) If S is not a reductive normal operator, then there exists a cyclic invariant subspace for S that has an open set of bounded point evaluations. (This open set can be constructed to be as large as possible.)

1. INTRODUCTION

Recently Brown [4] has shown that every subnormal operator S on a Hilbert space $\mathscr{H}$ has a nontrivial invariant subspace. By generalizing his duality results we shall show that every nonreductive subnormal operator has a rich supply of "analytic" invariant subspaces. There are enough of these subspaces that we are able to prove that every subnormal operator is reflexive. Furthermore our generalizations of Brown's duality results enable us to answer a question of Bram [3]. (Also consult [6].) We shall show that the weak-star closed algebra generated by S is precisely the algebra generated by S in the weak operator topology. (Furthermore the topologies are equivalent on this algebra.) If the reader's main interest lies in seeing the proof that subnormals are reflexive and/or lies in seeing how these analytic subspaces are constructed, then we suggest that he (or she) first read Section 2 through Lemma 6 and then proceed to Section 3. One may also find some satisfaction in skipping the proof of Lemma 6 on a first reading until one sees how it is used in the proof of Lemma 9. (Lemma 6 is also used to obtain Theorem 1.)

It seems appropriate to begin by summarizing Brown's proof of the invariant subspace theorem in order to motivate our duality results. Before we do this, it will be necessary to introduce some basic notions.

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Throughout this paper all Hilbert spaces are separable. The set of bounded linear operators on $\mathscr{K}$ is denoted $\mathscr{B}(\mathscr{K})$. The weak operator topology, abbreviated wot, is the topology defined by the seminorms

$$T \to \sum_{k=1}^{n} |\langle Tx_k, y_k\rangle|,$$

where $x_1, x_2, ..., x_n, y_1, ..., y_n$ vary through all finite subsets of $\mathscr{K}$. The σ-weak operator topology, abbreviated σ-wot, is defined by the seminorms

$$T \to \sum_{k=1}^{\infty} |\langle Tx_k, y_k\rangle|$$

where $\{x_k\}$ and $\{y_k\}$ vary over all sequences in $\mathscr{K}$ whose norms are square summable. Let $\mathscr{B}_1(\mathscr{K})$ denote the Banach space of trace-class operators on $\mathscr{K}$ with the trace norm. It is well known that the dual space of $\mathscr{B}_1(\mathscr{K})$ is $\mathscr{B}(\mathscr{K})$ via the action: if $T \in \mathscr{B}(\mathscr{K})$, then $L_T: \mathscr{B}_1(\mathscr{K}) \to \mathbb{C}$ is given by $L_T(B) = \mathrm{Tr}(TB)$ for all $B \in \mathscr{B}_1(\mathscr{K})$. (Here $\mathrm{Tr}(TB)$ denotes the trace of the operator TB.) It is a standard exercise to show the weak-star topology on $\mathscr{B}(\mathscr{K})$ is equivalent to the σ-weak topology.

If $T \in \mathscr{B}(\mathscr{K})$, let $\mathscr{O}\!\mathscr{l}(T)$ denote the σ-wot closure of the polynomials in T. Let $\mathscr{W}(T)$ denote the wot closure of $\mathscr{O}\!\mathscr{l}(T)$ and let $\mathscr{W}^*(T)$ denote the von Neumann algebra generated by T. Let $\mathscr{O}\!\mathscr{l}(T)_\perp$ denote the set $\{B \in \mathscr{B}_1(\mathscr{K}): \mathrm{Tr}(LB) = 0 \text{ for all } L \in \mathscr{O}\!\mathscr{l}(T)\}$. By the discussion above we see that $(\mathscr{O}\!\mathscr{l}(T), \sigma\text{-wot})^*$ is the Banach space $\mathscr{B}_1(\mathscr{K})/\mathscr{O}\!\mathscr{l}(T)_\perp$.

Let S be a subnormal operator on the space $\mathscr{H}$ with N its minimal normal extension on the space $\mathscr{K}$. The scalar spectral measure of N will always be denoted by μ. If $P^\infty(\mu)$ denotes the weak-star closure of the polynomials in $L^\infty(\mu)$ $(=L^1(\mu)^*)$, then $\mathscr{O}\!\mathscr{l}(N) = \{\varphi(N): \varphi \in P^\infty(\mu)\}$ and the map $\varphi \to \varphi(N)$ is an isometric isomorphism and weak-star homeomorphism. Since $\mathscr{O}\!\mathscr{l}(N)|_{\mathscr{H}} = \mathscr{O}\!\mathscr{l}(S)$, the map $\varphi = \varphi(S)$ from $P^\infty(\mu)$ onto $\mathscr{O}\!\mathscr{l}(S)$ is also an isometric isomorphism and weak-star homeomorphism [6].

We now are ready to outline Brown's remarkable proof. For this discussion only, let us say that a subnormal operator S has property B if

(i) the spectrum of S, denoted $\sigma(S)$, consists entirely of approximate point spectrum;

(ii) $P^\infty(\mu) = H^\infty$, where H^∞ is the classical Hardy space of bounded analytic functions on the open unit disc D;

(iii) $\|\varphi(S)\| = \|\varphi\|_{D\cap\sigma(S)}$ for all $\varphi \in H^\infty$. (Here, of course, $\|\ \|$ denotes the appropriate operator norm and supremum norm, respectively.)

Using the machinery in [6], one can verify that if every subnormal operator of type B has an invariant subspace, then every subnormal operator has one.

(The nontrivial part of the verification is why it is sufficient to consider those operators satisfying (iii). An argument based on the material of [6] can be found in [11].) Brown then handles these operators of type B via the following factorization theorem about the predual of $\mathfrak{A}(S)$.

BROWN'S THEOREM. *Let S be a subnormal operator of type B on $\mathcal{H}$. Let $L \in \mathcal{B}_1(\mathcal{H})/\mathfrak{A}(S)_\perp = (\mathfrak{A}(S), \sigma\text{-wot})^*$. Then there exist vectors a and b in $\mathcal{H}$ such that*

$$L(T) = \langle Ta, b\rangle$$

for every T in $\mathfrak{A}(S)$.

Using the fact that point evaluations in the open unit disc are weak-star continuous linear functionals, one can now easily produce invariant subspaces.

The work in this paper began with our efforts to determine which subnormal operators S, other than those of type B, have such a nice representation theorem for the predual of $\mathfrak{A}(S)$. The answer turns out to be all of them.

If X and Y are Banach spaces and $T\colon X^* \to Y^*$ is an isometric isomorphism that is also a weak-star homeomorphism, then the restriction of T^* to Y is an isometric isomorphism from Y onto X. Let

$$P^\infty(\mu)_\perp = \left\{ f \in L^1(\mu)\colon \int \varphi f \, d\mu = 0 \text{ for each } \varphi \in P^\infty(\mu) \right\}.$$

Using the last result and the natural maps described earlier, we see that $L^1(\mu)/P^\infty(\mu)_\perp$ $(\equiv P^\infty(\mu)_*)$ and $\mathcal{B}_1(\mathcal{H})/\mathfrak{A}(S)_\perp$ are isometrically isomorphic. (Recently, Ando [1] has shown that all preduals of H^∞ are isometrically isomorphic. Combining this fact with Grothendieck's result [8] that L^∞ has a unique predual, one can easily verify that $\mathfrak{A}(S)$ also does by using the methods in [6].) (The facts stated in this paragraph are not crucial to the results in this paper. All the arguments involving the predual of $\mathfrak{A}(S)$ in this paper could be done strictly in the function space.)

The following theorem is our generalization of Brown's representation theorem for the predual of $\mathfrak{A}(S)$.

THEOREM 1. *Let $L \in (\mathfrak{A}(S), \sigma\text{-wot})^*$. There exist vectors a and b in $\mathcal{H}$ and a universal constant C $(<2 \cdot 21^2)$ such that $\| a \| \leqslant C \| L \|^{1/2}$ and $\| b \| \leqslant C \| L \|^{1/2}$ and*

$$L(T) = \langle Ta, b\rangle \tag{1}$$

for each T in $\mathfrak{A}(S)$.

Remark. This theorem can be proved easily in the case where S is the unilateral shift. First, each L in $(\mathfrak{A}(S), \sigma\text{-wot})^*$ is represented by a function h in $L^1(m)$. (Here m denotes normalized Lebesgue measure on ∂D.) By Szegö's theorem there exists a function f in $H^2(m)$ such that $|f| \geqslant |h|^{1/2}$ almost everywhere. Defining g in $L^2(m)$ appropriately, we have $h = f\bar{g}$. Letting a equal f and b equal the projection of g into $H^2(m)$, we see that

$$L(T) = \langle Ta, b\rangle$$

for each T in $\mathscr{A}(S)$.

The proof of this theorem and related results constitute the major part of the second section of this paper.

Bram [3] has shown that every operator in $\mathscr{W}(S)$ is the restriction of an operator in $\mathscr{W}^*(N)$. He asks whether the latter operator is actually in $\mathscr{W}(N)$. Since $\mathscr{W}N) = \mathscr{A}(N)$ and $\mathscr{A}(N)|_{\mathscr{H}} = \mathscr{A}(S)$, this is equivalent to the condition $\mathscr{A}(S) = \mathscr{W}(S)$. The following theorem answers this question of Conway and Olin [6, Question 10.1]. It follows from Theorem 1 by using an idea of Sarason [15, Proposition 2.3].

THEOREM 2. *For every subnormal operator S, the mapping $i(T) = T$ is a homeomorphism from $(\mathscr{A}(S)$, σ-wot$)$ onto $(\mathscr{W}(S)$, wot$)$.*

Proof. We first show $\mathscr{A}(S)$ is wot closed. Suppose $\{T_j\}$ is a net in $\mathscr{A}(S)$ that converges weakly to the operator T. If $L \in (\mathscr{A}(S), \sigma\text{-wot})^*$, choose vectors a and b in $\mathscr{H}$ satisfying (1) of Theorem 1. It follows that

$$\Gamma(L) \equiv \langle Ta, b\rangle = \lim\langle T_j a, b\rangle$$

is a well-defined, bounded linear functional on $\mathscr{B}_1(\mathscr{H})/\mathscr{A}(S)_\perp$. Therefore, Γ is represented by some operator $f(S)$ in $\mathscr{A}(S)$.

For x, y in $\mathscr{H}$, let L_{xy} be defined on $\mathscr{A}(S)$ by

$$L_{xy}(R) = \langle Rx, y\rangle.$$

Clearly $L_{xy} \in (\mathscr{A}(S), \sigma\text{-wot})^*$, so

$$\Gamma(L_{xy}) = L_{xy}(f(S)) = \langle f(S)x, y\rangle.$$

But one can easily verify that $\Gamma(L_{xy}) = \langle Tx, y\rangle$, since x and y are arbitrary elements of $\mathscr{H}$, $T = f(S)$.

Because the wot topology is weaker that the σ-wot topology, clearly i is continuous. If $\{T_j\}$ is a net of operators converging wot to the operator T, then clearly by Theorem 1 and the fact that $\mathscr{A}(S) = \mathscr{W}(S)$, we have $L(T_j) \to L(T)$ for each L in $(\mathscr{A}(S), \sigma\text{-wot})^*$. Hence i has a continuous inverse.

Remark. The proof of Theorem 2 applies to any operator T such that the predual of $\mathscr{A}(T)$ has a representation like that of Theorem 1.

The final section of this paper is devoted to the proof of the following result.

THEOREM 3. *Every subnormal operator is reflexive.*

Recall that Lat T denotes the set of invariant subspaces for an operator T and Alg Lat T consists of those operators R such that Lat $R \supset$ Lat T. An operator T is reflexive if Alg Lat $T = \mathscr{W}(T)$. Theorem 3 demonstrates another property that subnormals inherit from their normal extensions [13]. The proof of this theorem is constructive in the following sense: it gives a concrete representation of a particular class of subspaces in Lat S that determine $\mathscr{W}(S)$. (See [2] for a related result.) If $P^\infty(\mu) = H^\infty$, then each of these determining subspaces can be thought of as analytic functions on the entire open unit disc.

2. The Predual of $P^\infty(\mu)$

We begin by considering a subnormal operator S on $\mathscr{H}$ whose minimal normal extension N is equal to multiplication by z, denoted M_z, on the space $L^2(\mu)$. We also assume that $P^\infty(\mu) = H^\infty$. (Throughout the paper we will draw heavily on the results in [6, 14] concerning the weak-star closure of the polynomials. However, a brief discussion seems in order as to what (the canonical case) $P^\infty(\mu) = H^\infty$ means. First of all, spt $\mu \subset \bar{D}$. If m denotes normalized Lebesgue measure on ∂D, the boundary of the open unit disc, then $\mu|_{\partial D}$ is absolutely continuous with respect to m. Furthermore, if $z \in D$ and $g \in H^\infty$, then $|g(z)| \leqslant \|g\|_\mu$, where $\|g\|_\mu$ denotes the essential supremum norm of g in $L^\infty(\mu)$. Hence, $\sup_{z\in D} |g(z)| = \|g\|_\mu$ for all $g \in H^\infty$. Finally, a net of polynomials $\{p_\alpha\}$ converges weak-star in $L^\infty(m)$ if and only if $\{p_\alpha\}$ converges weak-star in $L^\infty(\mu)$. That is, H^∞ and $P^\infty(\mu)$ have the same set of functions and their respective weak-star topologies are equivalent.) Since the operators of multiplication by z on two L^2-spaces are unitarily equivalent if and only if the measures are mutually absolutely continuous, we can assume there exists a Borel set $\Gamma \subset \partial D$ such that

$$m|_\Gamma = \mu|_{\partial D}.$$

Let X be any nonempty compact subset of ∂D such that $m(X\backslash\Gamma) = 0$.

Let $\{J_n\}$ denote the countable collection of pairwise disjoint open subarcs of ∂D such that

$$\partial D\backslash X = \bigcup J_n.$$

Let I_n denote the chord joining the endpoints of J_n. Let G denote the union of the regions bounded by the J_n's and I_n's. For $0 < r < 1$, let $A_r = \{z \in D: r < |z| < 1\}$ and let $G_r = G \cap A_r$.

Lemma 1. *Let ϵ, r belong to* (0, 1). *For each $e^{i\theta}$ in $\partial D\backslash X$ there exists $r_0 < 1$ ($r_0 = r_0(\epsilon, r, \theta)$) such that*

$$\tau(D\backslash G_r) < \epsilon$$

for each representing measure τ for evaluation of polynomials at $\rho e^{i\theta}$ for $1 > \rho \geqslant r_0$.

Proof. Without loss of generality we can assume $e^{i\theta} = 1$. Since the function $(1+z)/2$ peaks at the point 1, we can choose $r_0 < 1$ and $n > 0$ so that $(1+r_0)^n > 2^n(1 - \epsilon/2)$ and $|(1 + z)^n| < 2^{n-1}\epsilon$ on $D \backslash G_r$. Let ρ be any number such that $1 > \rho \geqslant r_0$ and τ a representing measure for the point ρ. We compute

$$1 - \frac{\epsilon}{2} < \left| \int \left(\frac{1+z}{2}\right)^n d\tau \right|$$

$$\leqslant \frac{\epsilon}{2}\tau(D\backslash G_r) + (1 - \tau(D\backslash G_r)).$$

It follows that $\tau(D\backslash G_r) < \epsilon/(2-\epsilon) < \epsilon$.

Remark. An examination of the proof above shows that on each compact subset K of J_n, there exists $r_0 = r_0(\epsilon, r, K)$.

For $s > 0$, let G_{rs} denote the set of those z in G_r such that $\tau(D\backslash G_r) < s$ for each representing measure τ for evaluation at z. For each λ in D let e_λ denote the weak-star continuous linear functional of evaluation at λ on $P^\infty(\mu)$. Let $\mathscr{B}^1(L^1(m|_\Gamma))$ denote the closed unit ball of $L^1(m|_\Gamma)$ and let

$$\mathscr{B} = \{ce_\lambda : |c| = 1, \lambda \in G_{rs} \cap \sigma(S)\} \cup \mathscr{B}^1(L^1(m|_\Gamma)).$$

Remark. We shall identify $L^1(\mu)$ functions in $P^\infty(\mu)_*$ via the quotient map. Note that $\mathscr{B}$ depends on r and s which we now fix.

LEMMA 2. *The closed convex hull of $\mathscr{B}$ equals the unit ball of $P^\infty(\mu)_*$.*

Proof. Suppose the result is false. Then by [12, Theorem 3.7] there exist φ in H^∞ and L_0 in the unit ball of $P^\infty(\mu)_*$ such that

$$|L_0(\varphi)| > 1 \geqslant \sup\{|L(\varphi)| : L \in \mathscr{B}\}. \tag{2.1}$$

The first inequality implies $\|\varphi\|_\infty > 1$. This in turn implies there exists a measurable subset F of ∂D with the following properties:

$$m(F) > 0, \tag{2.2}$$

$$|\varphi(e^{i\theta})| > 1 \qquad \text{for each } e^{i\theta} \text{ in } F, \tag{2.3}$$

$$\lim_{z \to e^{i\theta}} \varphi(z) = \varphi(e^{i\theta}) \qquad \text{for each } e^{i\theta} \text{ in } F, \tag{2.4}$$

where the limit is taken through any Stolz angle. If $m(F \cap \Gamma) > 0$, define $g \in L^1(m|_\Gamma)$ by

$$g(z) = \begin{cases} \dfrac{1}{m(\Gamma \cap F)} \dfrac{\bar{\varphi}(z)}{|\varphi(z)|}, & z \in F \cap \Gamma \\ 0, & z \in \Gamma \backslash F. \end{cases}$$

Clearly $g \in \mathscr{B}$, but $\int g\varphi \, d\mu > 1$, a contradiction to (2.1). Thus we may assume $m(F \cap \Gamma) = 0$. Consequently, $m(F \cap X) = 0$.

Hence we can assume there exist an integer n_0 and a compact set $K \subset J_{n_0} \cap F$ with $\mu(K) = 0$ while $m(K) > 0$. Without loss of generality, we assume $\lim_{h \downarrow 0}(m(K \cap C_h)/h) = 1$, where C_h denotes the open arc centered at 1 of length h. (See [9, Theorem 18.2].) By Szegö's theorem there exists a function ψ_1 in H^∞ with

$$| \psi_1 | = \begin{cases} 1 & \text{on} \quad K \\ \dfrac{1}{2 \| \varphi \|_\infty} & \text{on} \quad \partial D \backslash K \end{cases}$$

almost everywhere m. Note then the essential supremum norm of $\psi_1 \varphi$ (taken with respect to the measure $\mu |_{\partial D}$) is less than or equal to $\frac{1}{2}$.

Let W be a closed arc on ∂D with $K \subset W \subset J_{n_0}$. By Lemma 1, there exists $r_0 < 1$ such that $\rho e^{i\theta} \in G_{rs}$ for all $\rho \geqslant r_0$ and for all $e^{i\theta} \in W$. Since the function $(1 + z)/2$ peaks at 1, there exists a positive integer n_1 such that the function $\psi_2 \equiv ((1 + z)/2)^{n_1}$ has modulus less than $1/2 \| \varphi \|_\infty$ for each $z \in \bar{D}$ with z outside the region $R = \{\rho e^{i\theta} \colon r_0 \leqslant \rho \leqslant 1, e^{i\theta} \in W\}$. Let $\psi_3 = \varphi \psi_1 \psi_2$.

The function ψ_3 has the following properties:

$$\| \psi_3 \|_\infty > 1. \tag{2.5}$$

$$\text{For each } z \in D \backslash R \text{ we have } | \psi_3(z)| \leqslant \tfrac{1}{2}. \tag{2.6}$$

The essential sup norm (with respect to $\mu |_{\partial D}$) of ψ_3 is less than or equal to $\frac{1}{2}$. (2.7)

The essential supremum norm (with respect to $\mu |_R$) of ψ_3 is less than or equal to 1. (Use the second inequality of 2.1.) (2.8)

Combining (2.6)–(2.8), we see that essential supremum norm of ψ_3 (with respect to μ) is less than or equal to 1. Since $\| \psi_3 \|_\infty > 1$ and $P^\infty(\mu) = H^\infty$, this is a contradiction.

For elements a and b in $L^2(\mu)$, let $a \otimes b$ denote the element of $P^\infty(\mu)_*$ whose action on an element f of $P^\infty(\mu)$ is given by $\int fa\bar{b} \, d\mu$.

Suppose $T \in \mathscr{B}(\mathscr{H})$ and T is bounded below. If $0 \in \sigma(T)$, then

$$T^{n-1}\mathscr{H} \ominus T^n\mathscr{H} \neq \{0\} \tag{2.9}$$

for $n = 1, 2,\dots$. To see this, fix $n \geqslant 1$ and note that T^n is bounded below. Hence the map $x \to T^{n-1}x$, for $x \in \mathscr{H}$ induces a similarity between the operators T and $T |_{T^{n-1}\mathscr{H}}$. Therefore $0 \in \sigma(T |_{T^{n-1}\mathscr{H}})$.

Let $\| \cdot \|_*$ denote the quotient norm on $P^\infty(\mu)_*$.

LEMMA 3. *Let $\lambda \in \sigma(S) \cap D$ be such that $S - \lambda$ is bounded below. Let $v_n \in (S-\lambda)^{n-1}\mathscr{H} \ominus (S-\lambda)^n \mathscr{H}$ with $\| v_n \|_2 = 1$. Then*

$$| v_n |^2 \, d\mu \text{ represents evaluation at } \lambda \text{ for functions in } P^\infty(\mu). \tag{2.10}$$

$$v_n \to 0 \text{ weakly in } L^2(\mu). \tag{2.11}$$

$$\lim_{n\to\infty} \| a \otimes v_n \chi_B \|_* = 0 \text{ for every } a \in L^2(\mu) \text{ and for every Borel set } B \subset D. \tag{2.12}$$

$$\lim_{n\to\infty} \| v_n \otimes b \chi_B \|_* = 0 \text{ for every } b \in L^2(\mu) \text{ and for every Borel set } B \subset D. \tag{2.13}$$

Proof. (2.10) For each polynomial p we have

$$\begin{aligned}\int p \, | v_n |^2 \, d\mu &= \langle (p - p(\lambda)) \, v_n , v_n \rangle + p(\lambda) \langle v_n , v_n \rangle \\ &= 0 + p(\lambda).\end{aligned}$$

Taking weak-star limits, we obtain the conclusion.

(2.11) Observe that $\{v_n\}$ is an orthonormal sequence.

(2.12) Let $\epsilon > 0$. Let $B_r = B \cap A_r$. A routine argument from measure theory shows there exists $r_0 < 1$ such that $\| a \chi_{B_{r_0}} \| < \epsilon$. Let $F = B \backslash B_{r_0}$. It is easy to find a positive integer M_1 such that $| \sum_{M_1}^\infty c_m z^m | < \epsilon / \| a \|_2$ on F if $\Sigma \, | c_m |^2 \leqslant 1$. Since $v_n \to 0$ weakly in $L^2(\mu)$, there exists a positive integer M_2 such that $|\langle z^m a \chi_F , v_n \rangle| < \epsilon / M_1$ for $m = 1, \ldots, M_1$ and $n \geqslant M_2$.

Let $f = \Sigma c_m z^m$ belong to $P^\infty(\mu)$ with $\| f \|_\infty \leqslant 1$. Then for $n \geqslant M_2$,

$$\begin{aligned}\left| \int f a \bar{v}_n \chi_B \, d\mu \right| &\leqslant \left| \int_{B_{r_0}} f a \bar{v}_n \, d\mu \right| + \left| \sum_{m=1}^{M_1 - 1} c_m \int_F z^m a \bar{v}_n \, d\mu \right| \\ &\quad + \left| \int_F \left(\sum_{M_1}^\infty c_m z^m \right) a \bar{v}_n \, d\mu \right| \\ &\leqslant \| f \|_\infty \| a \chi_{B_{r_0}} \|_2 \| v_n \|_2 + M_1 \frac{\epsilon}{M_1} + \frac{\epsilon}{\| a \|_2} \| a \|_2 \| v_n \|_2 \leqslant 3\epsilon.\end{aligned}$$

(2.13) The proof is similar to that of (2.12).

LEMMA 4. *Let $\lambda \in \sigma(S) \cap D$ be such that $S - \lambda$ is not bounded below. Let $\{x_n\}$ be a sequence in $\mathscr{H}$ with $\| x_n \|_2 = 1$ for all n and such that* $\lim_{n\to\infty} \|(S-\lambda) x_n\|_2 = 0$. *Then we have*

$$\lim_{n\to\infty} \| e_\lambda - x_n \otimes x_n \|_* = 0. \tag{2.14}$$

$$\lim_{n\to\infty} \| x_n b \|_1 = 0 \qquad \text{for every } b \in L^2(\mu) \text{ such that } b(\lambda) = 0. \tag{2.15}$$

If W is an open subset of D with $\lambda \in W$

$$\text{then} \qquad \lim_{n\to\infty} \int_{\bar{D}\setminus W} |x_n|^2 \, d\mu = 0. \tag{2.16}$$

Furthermore, if $\beta \in \sigma(S) \cap D$ with $\beta \neq \lambda$ and $\{y_n\}$ is a sequence in $\mathscr{H}$ with $\|y_n\|_2 = 1$ for all n such that $\lim_{n\to\infty}\|(S-\beta)y_n\|_2 = 0$, *then*

$$\lim_{n\to\infty} \|x_n y_n\|_1 = 0. \tag{2.17}$$

Proof. (2.14) Let $f \in P^\infty(\mu)$ with $\|f\|_\infty \leqslant 1$. Using the maximum modulus principle, we have

$$\begin{aligned}
\left| \int f |x_n|^2 \, d\mu - e_\lambda(f) \right| &= \left| \int (f - f(\lambda)) |x_n|^2 \, d\mu \right| \\
&\leqslant \frac{2}{1-|\lambda|} \int |z-\lambda| \, |x_n|^2 \, d\mu \\
&\leqslant \frac{2}{1-|\lambda|} \|(z-\lambda) x_n\|_2 \|x_n\|_2 \\
&= \frac{2}{1-|\lambda|} \|(S-\lambda) x_n\|_2 .
\end{aligned}$$

(2.15) Let $\epsilon > 0$. Since $b(\lambda) = 0$ and the measure $|b|^2 \, d\mu$ is regular, there exists a neighborhood N_λ of λ such that $\int_{N_\lambda} |b|^2 \, d\mu < \epsilon$. We compute:

$$\begin{aligned}
\int |x_n b| \, d\mu &= \int_{N_\lambda} |x_n b| \, d\mu + \int_{\bar{D}\setminus N_\lambda} |x_n b| \, d\mu \\
&\leqslant \|x_n\|_2 \|b\chi_{N_\lambda}\|_2 + \|x_n \chi_{\bar{D}\setminus N_\lambda}\|_2 \|b\|_2 \\
&< \epsilon + \|x_n \chi_{\bar{D}\setminus N_\lambda}\|_2 \|b\|_2 .
\end{aligned}$$

For all n sufficiently large, we have $\|x_n \chi_{\bar{D}\setminus N_\lambda}\|_2 \|b\|_2 \leqslant \epsilon$. Otherwise, for infinitely many n, $\|(S-\lambda)x_n\|_2$ is greater than $\epsilon/\|b\|_2$ times the distance from λ to $\bar{D}\setminus N_\lambda$, a contradiction.

(2.16) This follows from an argument similar to the one used in the last part of the proof of (2.15).

(2.17) Let U and V be disjoint open discs contained in D with $\lambda \in U$ and $\beta \in V$. Then

$$\begin{aligned}
\left| \int |x_n y_n| \, d\mu \right| &\leqslant \int_{\bar{D}\setminus U} |x_n y_n| \, d\mu + \int_{\bar{D}\setminus V} |x_n y_n| \, d\mu \\
&\leqslant \|x_n \chi_{\bar{D}\setminus U}\|_2 \|y_n\|_2 + \|x_n\|_2 \|y_n \chi_{\bar{D}\setminus V}\|_2 .
\end{aligned}$$

We see that each of these last summands goes to zero as $n \to \infty$ from (2.16).

LEMMA 5. *There exists a vector $f \in \mathscr{H}$ such that $|f| > 0$ almost everywhere μ. For each vector $h \in L^2(\mu|_\Gamma)$* and $\epsilon > 0$ there exists $g \in \mathscr{H}$ such that

$$|h| + \epsilon \geqslant |g| \geqslant |h|$$

almost everywhere (m) on Γ.

Proof. Using a result of Chaumat [5, Proposition 4, Chap. 1], we see there exists $f \in \mathscr{H}$ with $\|f\|_2 \leqslant 1$ such that $kd\mu \ll fd\mu$ for every k in $\mathscr{H}$. Since M_z on $L^2(\mu)$ is the minimal normal extension of S, clearly $|f| > 0$ almost everywhere (μ). To prove the rest of the lemma we may assume $h \geqslant 0$.

Define a countable collection of sets $\Gamma_{i,j}$ as follows: for $i \geqslant 0$ let

$$\Gamma_{i0} = \{z \in \Gamma: 1 - i < h \leqslant i,\ 1 \leqslant |f|\}$$

and for $i \geqslant 0, j \geqslant 1$ we let

$$\Gamma_{ij} = \{z \in \Gamma: 1 - i < h \leqslant i,\ 1/2^j \leqslant |f| < 1/2^{j-1}\}.$$

Clearly the Γ_{ij}'s are pairwise disjoint and $\bigcup_{i,j} \Gamma_{ij} = \Gamma$. By Szegö's theorem for each (i, j) there exists $\varphi_{ij} \in H^\infty$ such that

$$|\varphi_{ij}| = \begin{cases} \dfrac{h + \epsilon}{|f|} & z \in \Gamma_{ij} \\ \min\left\{\dfrac{\epsilon}{2^i 2^j}, \dfrac{\epsilon}{2^i 2^j |f|}\right\} & z \in \Gamma \backslash \Gamma_{ij}. \end{cases}$$

Multiplying φ_{ij} by an appropriate power of z, we can assume $\|\varphi_{ij} f \chi_D\| < 1/2^i 2^j$.

Clearly $\sum_j \sum_i \|\varphi_{ij} f\|_2^2 < \infty$. Hence $\sum_{i,j} \varphi_{ij} f = \sum_j \sum_i \varphi_{ij} f$ converges in $\mathscr{H}$, say to the vector g. If $z \in \Gamma_{ij}$, then

$$\begin{aligned} |g(z)| &\geqslant |\varphi_{ij} f(z)| - \sum_{(l,p) \neq (i,j)} |\varphi_{lp} f(z)| \\ &\geqslant (h(z) + \epsilon) - \left(\sum_{l=1} \sum_{p=1} \frac{\epsilon}{2^l 2^p}\right) \\ &= h(z). \end{aligned}$$

A similar computation shows $|g(z)| \leqslant h(z) + 2\epsilon$.

Convention. Many times in the rest of this paper we will end up with estimates of the form $f \geqslant Cg$, where f and g are functions and C is some positive constant depending on f and g. It then follows that g/f is a bounded function if we define $0/0$ to be zero.

LEMMA 6. *Let $L \in P^{\infty}(\mu)_*$ and let $d_1, \ldots, d_i$ be functions in $L^2(\mu)$. Let δ, ϵ, ϵ', r belong to* (0, 1) *with $\epsilon < \frac{1}{10}$ and $\delta < \frac{1}{16}$. Suppose there exist functions a and b in $L^2(\mu)$ such that*

$$\|L - a \otimes b\|_* < \delta^4. \tag{2.18}$$

Then there exist functions $x \in \mathscr{H}$ and $y \in L^2(\mu\,|_{G_r \cup \Gamma})$ with the following properties:

$$\|x\|_2 < 17\delta, \tag{2.19}$$

$$\|b + y\|_2 < \frac{1}{1 - 8\delta}\|b\|_2 + 8\delta^2, \tag{2.20}$$

$$|a + x| \geqslant (1 - 8\delta)\,|a| \tag{2.21}$$

almost everywhere on Γ,

$$\|L - (a + x) \otimes (b + y)\|_* < \epsilon, \tag{2.22}$$

and

$$\|x \otimes d_k \chi_D\|_* < \epsilon' \tag{2.23}$$

for $k = 1, 2, \ldots, i$.

Proof. Clearly if $\rho > r$ and $y \in L^2(\mu\,|_{G_\rho \cup \Gamma})$, then $y \in L^2(\mu\,|_{G_r \cup \Gamma})$. Observe that for any function $g \in L^1(\mu)$ that $\lim_{\rho \to 1} \|g\chi_{A_\rho}\|_1 = 0$. Thus, for all ρ sufficiently close to 1, it follows that

$$\|L - (a\chi_{\bar{D}\setminus A_\rho} \otimes b\chi_{\bar{D}\setminus A_\rho}\|_* < \delta^4.$$

Therefore if there exist functions $x \in \mathscr{H}$ and $y \in L^2(\mu\,|_{G_\rho \cup \Gamma})$ for all sufficiently large ρ (x and y depend on ρ) with the properties

$$\|x\|_2 < 17\delta, \tag{2.19$'$}$$

$$\|b\chi_{\bar{D}\setminus A_\rho} + y\|_2 < \frac{1}{1 - 8\delta}\|b\chi_{\bar{D}\setminus A_\rho}\|_2 + 8\delta^2, \tag{2.20$'$}$$

$$|a + x| \geqslant (1 - 8\delta)\,|a| \tag{2.21$'$}$$

almost everywhere on Γ,

$$\|L - (a\chi_{\bar{D}\setminus A_\rho} + x) \otimes (b\chi_{\bar{D}\setminus A_\rho} + y)\|_* < \epsilon/2, \tag{2.22$'$}$$

$$\|x \otimes d_k\chi_{\bar{D}\setminus A_\rho}\|_* < \epsilon'/2 \qquad \text{for all } k = 1, 2, \ldots, i, \tag{2.22$'$}$$

then we can find functions $x \in \mathscr{H}$ and $y \in L^2(\mu\,|_{G_r \cup \Gamma})$ satisfying (2.19)–(2.23). Hence, we can assume that

$$a\chi_{A_r} = b\chi_{A_r} = 0 \qquad \text{almost everywhere } \mu, \tag{2.24}$$

$$d_k\chi_{A_r} = 0 \qquad \text{almost everywhere } \mu. \tag{2.25}$$

Let $L_1 = L - a \otimes b$. Applying Lemma 2 with $s = \frac{1}{6}\epsilon$, we obtain $g \in L^1(m|_\Gamma)$ and finite sequences $\{\lambda_m\}_1^{M_2}$, $\{\gamma_m\}_1^{M_2}$ and $\{c_m\}_1^{M_2}$ so that

$$\lambda_m \in \sigma(S) \cap G_{rs}, \tag{2.26}$$

$$c_m > 0, \tag{2.27}$$

$$\|g\|_1 + \sum c_m^2 < \delta^4, \tag{2.28}$$

$$|\gamma_m| = 1, \tag{2.29}$$

$$\left\| L_1 - \left(\sum \gamma_m c_m^2 e_{\lambda_m} + g \right) \right\|_* < \tfrac{1}{6}\epsilon. \tag{2.30}$$

Order the λ_m's so that $S - \lambda_m$ is bounded below for $m = 1,\ldots, M_1$ and $S - \lambda_m$ is not bounded below for $m = M_1 + 1,\ldots, M_2$.

We start by approximating the term $\sum_1^{M_1} \gamma_m c_m^2 e_{\lambda_m}$. Let δ_1 be a small positive number (to be determined later). Let v_{mn} be an element of $(z - \lambda_m)^{n-1}\mathscr{H} \ominus (z - \lambda_m)^n \mathscr{H}$ with $\|v_{mn}\|_2 = 1$. We now define $v_1, \ldots, v_{M_1}$ by induction. Use Lemma 3 to choose n so that

$$c_1 \| v_{1n} \otimes b\chi_D \|_* < \delta_1, \tag{2.31}$$

$$|v_{1n}(\lambda_j)| < \delta_1 \qquad \text{for all} \quad j > M_1, \tag{2.32}$$

$$c_1 \| v_{1n} \otimes d_k \chi_D \|_* < \delta_1 \qquad \text{for} \quad k = 1, 2,\ldots, i. \tag{2.33}$$

Let $v_1 = v_{1n}$. Suppose $v_1, \ldots, v_{m-1}$ have been defined. Use Lemma 3 to choose n so that:

$$c_m \| v_{mn} \otimes b\chi_D \|_* < \delta_1, \tag{2.34}$$

$$c_m c_p \| v_{mn} \otimes v_p \chi_{G_r} \|_* < \delta_1 \qquad \text{for all } p = 1,\ldots, m-1, \tag{2.35}$$

$$c_m c_p \| v_p \otimes v_{mn} \chi_{G_r} \|_* < \delta_1 \qquad \text{for all } p = 1,\ldots, m-1, \tag{2.36}$$

$$c_m c_p |\langle v_p, v_{mn} \rangle| < \delta_1 \qquad \text{for all } p = 1,\ldots, m-1, \tag{2.37}$$

$$|v_{mn}(\lambda_j)| < \delta_1 \qquad \text{for all } j > M_1, \tag{2.38}$$

$$c_m \| v_{mn} \otimes d_k \chi_D \|_* < \delta_1 \qquad \text{for all } k = 1,\ldots, i. \tag{2.39}$$

Let $v_m = v_{mn}$. Let $x_1 = \sum_1^{M_1} \gamma_m c_m v_m$ and $y_1 = \sum_1^{M_1} c_m v_m$. Note that $\|x_1\|_2^2 < \delta^4 + M_1^2 \delta_1$, $\|y_1\|_2^2 < \delta^4 + M_1^2\delta_1$, and $\|x_1 \otimes d_k \chi_D\|_* < M_1 \delta_1$ for all $k = 1,\ldots, i$.

We now approximate the term $\sum_{M_1+1}^{M_2} \gamma_m c_m^2 e_{\lambda_m}$. Let δ_2 be a small positive constant (to be determined later). For each m, $M_1 + 1 \leqslant m \leqslant M_2$, let $\{w_{mn}\}$ be a sequence of unit vectors in $\mathscr{H}$ such that $\|(S - \lambda_m) w_{mn}\|_2 \to 0$ as $n \to \infty$. By (2.14) and (2.16) of Lemma 4, there exists an integer n_1 such that for every m and for every $n \geqslant n_1$, we have

$$\| e_{\lambda_m} - w_{mn} \otimes w_{mn} \chi_{G_r} \|_* < \tfrac{1}{6}\epsilon. \tag{2.40}$$

By (2.24) and (2.15) there exists an integer n_2 so that for every m and for every $n \geqslant n_2$, we have

$$c_m \| w_{mn} a \|_1 < \delta_2 , \tag{2.41}$$

$$c_m \| w_{mn} b \|_1 < \delta_2 , \tag{2.42}$$

[and also using (2.38)] we have

$$c_p c_q \| w_{qn} v_p \|_1 < \delta_2 + \delta_1 \tag{2.43}$$

for each p and q. Using (2.17), we see there exists an integer n_3 such that for every $n \geqslant n_3$ we have

$$c_p c_q \| w_{pn} w_{qn} \|_1 < \delta_2 \qquad \text{for every} \quad p \neq q. \tag{2.44}$$

Using (2.15) and (2.25), we see there exists an integer n_4 such that for every $n \geqslant n_4$,

$$c_m \| w_{mn} \otimes d_k \chi_D \|_* < \delta_2 \tag{2.45}$$

for all m and for all $k = 1,\ldots, i$. Let $n = \max\{n_1, n_2, n_3, n_4\}$ and let $w_m = w_{mn}$ for $M_1 + 1 \leqslant m \leqslant M_2$. Let $x_2 = \sum_{M_1+1}^{M_2} \gamma_m c_m w_m$ and let $y_2 = \sum_{M_1+1}^{M_2} c_m w_m$. Note that $\| x_2 \|_2^2 < \delta^4 + (M_2 - M_1)^2 \delta_2$, $\| y_2 \|_2^2 < \delta^4 + (M_2 - M_1)^2 \delta_2$, and $\| x_2 \otimes d_k \chi_D \|_* < (M_2 - M_1) \delta_2$ for all $k = 1,\ldots, i$.

Let $a_1 = a + x_1 + x_2$ and $b_1 = b\chi_D + (y_1 + y_2)\chi_{G_r}$. We also set $h = a\bar{b}\chi_\Gamma + \sum_1^{M_1} \gamma_m c_m{}^2 | v_m |^2 \chi_\Gamma$. We compute, using (2.10), (2.24), and inequalities (2.31) through (2.44):

$$\begin{aligned}
&\| L - (a_1 \otimes b_1 + h + g) \|_* \\
&\quad = \Big\| L - \Big[a \otimes b\chi_D + a \otimes y_1 \chi_{G_r} + a \otimes y_2 \chi_{G_r} \\
&\qquad + x_1 \otimes b\chi_D + \sum_1^{M_1} \gamma_m c_m{}^2 v_m \otimes v_m \chi_{G_r} \\
&\qquad + \sum_{p=1}^{M_1} \sum_{q \neq p} \gamma_p c_p c_q v_p \otimes v_q \chi_{G_r} + x_1 \otimes y_2 \chi_{G_r} \\
&\qquad + x_2 \otimes b\chi_D + x_2 \otimes y_1 \chi_{G_r} + \sum_{M_1+1}^{M_2} \gamma_m c_m{}^2 w_m \otimes w_m \chi_{G_r} \\
&\qquad + \sum_{p=M_1+1}^{M_2} \sum_{q \neq p} \gamma_p c_p c_q w_p \otimes w_q \chi_{G_r} \\
&\qquad + a \otimes b\chi_\Gamma + \sum_1^{M_1} \gamma_m c_m{}^2 v_m \otimes v_m \chi_\Gamma + g \Big] \Big\|_*
\end{aligned}$$

$$\leqslant \left\| L_1 - \left[\sum_1^{M_2} \gamma_m c_m{}^2 e_{\lambda_m} + g \right] \right\|_*$$

$$+ \left\| \sum_1^{M_2} \gamma_m c_m{}^2 e_{\lambda_m} - \sum_1^{M_1} \gamma_m c_m{}^2 \mid v_m \mid^2 \chi_{G_r \cup \Gamma} - \sum_{M_1+1}^{M_2} \gamma_m c_m{}^2 w_m \otimes w_m \chi_{G_r} \right\|_*$$

$$+ O(\delta_1) + O(\delta_2)$$

$$\leqslant \tfrac{1}{6}\epsilon + \sum_1^{M_1} c_m{}^2 [\| e_{\lambda_m} - \mid v_m \mid^2 \|_* + \| \mid v_m \mid^2 \chi_{D \setminus G_r} \|_*]$$

$$+ \sum_{M_1+1}^{M_2} c_m{}^2 \| e_{\lambda_m} - w_m \otimes w_m \chi_{G_r} \|_* + O(\delta_1) + O(\delta_2)$$

$$\leqslant \tfrac{1}{6}\epsilon + \left(\sum_1^{M_1} c_m{}^2 \right) (\tfrac{1}{6}\epsilon) + \left(\sum_{M_1+1}^{M_2} c_m{}^2 \right) (\tfrac{1}{6}\epsilon) + O(\delta_1) + O(\delta_2)$$

$$\leqslant \tfrac{1}{3}\epsilon + O(\delta_1) + O(\delta_2).$$

(In the next to the last estimate keep in mind the result of Lemma 1.) We also have (by (2.39) and (2.45)) that $\|(x_1 + x_2) \otimes d_k \chi_D \|_* \leqslant M_1 \delta_1 + M_2 \delta_2$ for all $k = 1,\ldots, i$. We now choose δ_1 and δ_2 so small that each of the terms $\| x_1 \|_2$, $\| y_1 \|_2$, $\| x_2 \|_2$, and $\| y_2 \|_2$ is bounded by δ^2 and small enough so that

$$\|L - (a_1 \otimes b_1 + h + g)\|_* < \tfrac{1}{2}\epsilon, \tag{2.46}$$

and for all $k = 1,\ldots, i$,

$$\|(x_1 + x_2) \otimes d_k \chi_D \|_* < \tfrac{1}{5}\epsilon'. \tag{2.47}$$

Using Lemma 5 and multiplying by an appropriate power of z, we see there exists a function $f \in \mathscr{H}$ such that $|f| \geqslant |g|^{1/2}$ almost everywhere on Γ and $\|f\|_2 \leqslant \delta^2$. Let $g_1 = f$ and $g_2 = (\bar{g}/\bar{f})\,\chi_\Gamma$. Then

$$g = g_1 \bar{g}_2 \tag{2.48}$$

and $\| g_1 \|_2$ and $\| g_2 \|_2$ are bounded by δ^2.

We next construct $x \in \mathscr{H}$ and $y \in L^2(\mu\,|_{G_r \cup \Gamma})$ so that

$$(a + x)\overline{(b + y)}\,\chi_\Gamma = g + h, \tag{2.49}$$

$$\|(a + x) \otimes (b + y)\,\chi_D - a_1 \otimes b_1 \|_* < \tfrac{1}{2}\epsilon. \tag{2.50}$$

First we deal with the function h_1 defined by $h_1 = \sum_1^{M_1} \gamma_m c_m{}^2 \mid v_m \mid^2 \chi_\Gamma$. Let $h_2 = \sum_1^{M_1} c_m{}^2 \mid v_m \mid^2 \chi_\Gamma$ and let δ_3 be a small positive constant (to be determined later). Let $V = \{z \in \Gamma : h_2(z) > 0\}$. We define a collection $\{V_m\}_1^{M_1}$ of sets by induction. Let

$$V_1 = \{z \in V : |v_1(z)| \geqslant |v_i(z)| \quad \text{for all} \quad i \geqslant 1\}.$$

For $m > 1$, let

$$V_m = \{z \in V \backslash (V_1 \cup \cdots \cup V_{m-1}) : |v_m(z)| \geqslant |v_i(z)| \text{ for all } i \geqslant 1\}.$$

Since $\sum c_m{}^2 < 1$, it follows that $|v_m|^2 \geqslant h_2$ on V_m. We decompose each V_m into V_{m1} and V_{m2} as follows:

$$V_{m1} = \{z \in V_m : \delta_3{}^2 |v_m(z)|^2 \leqslant h_2(z)\}$$

and

$$V_{m2} = V_m \backslash V_{m1}.$$

By Szegö's theorem there exists a function φ_m in H^∞ such that

$$|\varphi_m(z)| = \begin{cases} \dfrac{(h_2(z))^{1/2}}{|v_m(z)|} & z \in V_{m1} \\ \delta_3 & z \in V_{m2} \\ \delta_3{}^2 & z \in \partial D \backslash V_m . \end{cases}$$

Multiplying φ_m by an appropriate power of z, we may assume $\| \varphi_m v_m \chi_D \|_2 < \delta_3$. Let $v = \sum_1^{M_1} \varphi_m v_m$. We are interested in its norm:

$$\| v \|_2 \leqslant \left\| \sum \varphi_m v_m \chi_D \right\|_2 + \left\| \sum \varphi_m v_m \chi_{\bigcup V_i} \right\|_2$$

$$\left(\text{note: } \mu\left(V \Big\backslash \left(\bigcup_1^{M_1} V_i\right)\right) = 0\right)$$

$$\begin{aligned} &< M_1 \delta_3 + \left\| \sum \varphi_m v_m \chi_{\bigcup_{i \neq m} V_i} \right\|_2 \\ &\quad + \left\| \sum \varphi_m v_m \chi_{V_m} \right\|_2 \\ &\leqslant M_1 \delta_3 + M_1 \delta_3{}^2 + \left(\int_\Gamma h_2 \, d\mu \right)^{1/2} \\ &\leqslant 2 M_1 \delta_3 + \delta^2. \end{aligned}$$

We now show that $|v|^2 \geqslant \frac{1}{2} h_2$ on Γ if δ_3 is chosen sufficiently small. On $\Gamma \backslash V$ both functions equal zero a.e. (μ). On V_{m1} we have

$$\begin{aligned} |v| &\geqslant |\varphi_m v_m| - \sum_{i \neq m} |\varphi_i v_i| \\ &\geqslant (h_2)^{1/2} - \delta_3{}^2 \sum |v_i| \\ &\geqslant (h_2)^{1/2} - M_1 \delta_3 (h_2)^{1/2}. \end{aligned}$$

On V_{m2} we have

$$\begin{aligned} |v| &\geqslant |\varphi_m v_m| - \sum_{i \neq m} |\varphi_i v_i| \\ &\geqslant \delta_3 |v_m| - M_1 \delta_3^2 |v_m| \\ &= \delta_3(1 - M_1\delta_3) |v_m| \\ &\geqslant (1 - M_1\delta_3)(h_2)^{1/2}. \end{aligned}$$

We now choose δ_3 sufficiently small so that $\| v \|_2 < \delta^2$ and $| v |^2 \geqslant \frac{1}{2} h_2$ on Γ. Hence $| v |^2 \geqslant \frac{1}{2} | h_1 |$ on Γ.

We define the sets A_1, A_2, A_3, and A_4 as follows:

$$\begin{aligned} A_1 &= \{z \in \Gamma : |x_1| \geqslant \delta |a|, |x_1| \geqslant |x_2|, |x_1| \geqslant |v|, |x_1| \geqslant |g_1|\}, \\ A_2 &= \{z \in \Gamma : |x_2| \geqslant \delta |a|, |x_2| > |x_1|, |x_2| \geqslant |v|, |x_2| \geqslant |g_1|\}, \\ A_3 &= \{z \in \Gamma : |v| \geqslant \delta |a|, |v| > |x_1|, |v| > |x_2|, |v| \geqslant |g_1|\}, \\ A_4 &= \{z \in \Gamma : |g_1| \geqslant \delta |a|, |g_1| > |x_1|, |g_1| > |x_2|, |g_1| > |v|\}. \end{aligned}$$

Let $f_j \in H^\infty$, $j = 1,\ldots, 4$, be chosen so that

$$|f_j| = \begin{cases} 4/\delta, & z \in A_j \\ 1, & z \in \partial D \backslash A_j \end{cases}$$

for $j = 1, 2, 3, 4$. Multiplying by an appropriate power of z we may assume $\|f_1 x_1 \bar{b}_1 \chi_D\|_1$, $\|f_2 x_2 \bar{b}_1 \chi_D\|_1$, $\|f_3 v \bar{b}_1 \chi_D\|_1$, and $\|f_4 g_1 \bar{b}_1 \chi_D\|_1$ are each less than $\frac{1}{8}\epsilon$; and that $\|f_1 x_1 \bar{d}_k \chi_D\|_1$, $\|f_2 x_2 \bar{d}_k \chi_D\|_1$, $\|f_3 v \bar{d}_k \chi_D\|_1$, and $\|f_4 g_1 \bar{d}_k \chi_D\|_1$ are each less than $\frac{1}{8}\epsilon'$ for $k = 1,\ldots, i$.

Now let $x = x_1 + x_2 + f_1 x_1 + f_2 x_2 + f_3 v + f_4 g_1$. Observe that $x \in \mathscr{H}$ and by (2.47) and the last paragraph that

$$\| x \otimes d_k \chi_D \| < \epsilon' \tag{2.51}$$

for $k = 1,\ldots, i$. We are interested in the modulus of $a + x$. On A_1, we have (remember $\delta < \frac{1}{16}$)

$$\begin{aligned} |a + x| &\geqslant |f_1 x_1| - (|a| + |x_1| + |x_2| + |f_2 x_2| + |f_3 v| + |f_4 g_1|) \\ &\geqslant \frac{4}{\delta} |x_1| - \left(\frac{|x_1|}{\delta} + 5 |x_1|\right) \\ &\geqslant \frac{2}{\delta} |x_1| \\ &\geqslant |a| + |v| + |g_1|. \end{aligned} \tag{2.52}$$

Similarly on A_2, A_3, and A_4 we have

$$|a + x| \geqslant |a| + |v| + |g_1|. \tag{2.53}$$

But we also have on $\Gamma\backslash(A_1 \cup A_2 \cup A_3 \cup A_4)$ that

$$\begin{aligned} | a + x | &\geqslant | a | + | v | + | g | - (| x_1 | + | x_2 | + | f_1 x_1 | + | f_2 x_2 | + | f_3 v | \\ &\qquad + | f_4 g_1 | + | v | + | g_1 |) \\ &\geqslant (| a | + | v | + | g_1 |)(1 - 8\delta). \end{aligned} \tag{2.54}$$

Combining inequalities (2.52), (2.53), and (2.54), we have

$$| a + x | \geqslant (| a | + | v | + | g_1 |)(1 - 8\delta) \tag{2.55}$$

almost everywhere (m) on Γ. We also note that

$$\begin{aligned} \| x \|_2 &\leqslant \| x_1 \|_2 + \| x_2 \|_2 + \| f_1 x_1 \|_2 + \| f_2 x_2 \|_2 + \| f_3 v \|_2 + \| f_4 g_1 \|_2 \\ &\leqslant 2\delta^2 + 16\delta \\ &\leqslant 17\delta. \end{aligned} \tag{2.56}$$

It is easy to see that there exist functions ψ_1 and ψ_2 in $L^\infty(\mu \mid_\Gamma)$ with $| \psi_1 | = | \psi_2 | = 1$ almost everywhere on Γ such that, on Γ, we have

$$| b + 2\psi_1 v + \psi_2 g_2 | = | b | + 2 | v | + | g_2 |.$$

Let $b_2 = b + 2\psi_1 v + \psi_2 g_2$. Using (2.48), (2.55), and the fact $| v |^2 \geqslant \frac{1}{2} h_1$ on Γ, we have

$$\begin{aligned} |(a + x)\, b_2 | &\geqslant (1 - 8\delta)(| a | + | v | + | g_1 |)(| b | + 2 | v | + | g_2 |) \\ &\geqslant (1 - 8\delta) \left| a\bar{b} + \sum_1^{M_1} \gamma_m c_m{}^2 | v_m |^2 + g \right| \end{aligned}$$

a.e. on Γ. Therefore, the function θ defined as

$$\theta = \frac{a\bar{b} + \sum \gamma_m c_m{}^2 | v_m |^2 + g}{(a + x)\, b_2} \chi_\Gamma$$

belongs to $L^\infty(m)$. We now define $y \in L^2(\mu_{G_r \cup \Gamma})$ as follows:

$$y = \overline{(\theta b_2} - b)\, \chi_\Gamma + (y_1 + y_2)\, \chi_{G_r}.$$

Note that $\overline{b + y} = \theta b_2 \chi_\Gamma + \bar{b}_1$. We estimate the norm of $b + y$:

$$\begin{aligned} \| b + y \|_2 &= \| \theta(b + 2\psi_1 v + \psi_2 g_2)\, \chi_\Gamma + \bar{b}\chi_D + (\bar{y}_1 + \bar{y}_2)\, \chi_{G_r} \|_2 \\ &\leqslant \frac{1}{1 - 8\delta} (\| b \|_2 + 3\delta^2) + 2\delta^2 \\ &\leqslant \frac{1}{1 - 8\delta} \| b \|_2 + 8\delta^2. \end{aligned} \tag{2.57}$$

(The last inequality follows because $\delta < \frac{1}{16}$.)

Now on Γ we have

$$(a + x)(\overline{b + y}) = a\bar{b} + \sum \gamma_m c_m{}^2 \mid v_m \mid^2 + g$$

which establishes (2.49). On the open disc D

$$(a + x)(\overline{b + y}) = (a + x)\, \bar{b}_1 .$$

Therefore we have

$$\begin{aligned}&\|(a + x)(\overline{b + y})\, \chi_D - a_1 \otimes b_1 \|_* \\ &\qquad = \| f_1 x_1 \otimes b_1 + f_2 x_2 \otimes b_1 + f_3 v \otimes b_1 + f_4 g_1 \otimes b_1 \|_* < \tfrac{4}{8}\epsilon\end{aligned}$$

which establishes (2.50). Thus, by (2.46), (2.49), and (2.50), we have

$$\| L - (a + x) \otimes (b + y) \|_* < \epsilon. \tag{2.58}$$

The results of (2.51), (2.55), (2.56), (2.57), and (2.58) are precisely the conclusions of the lemma.

LEMMA 7. *Let $L \in P^\infty(\mu)_*$ and $r < 1$. Then there exist vectors $x \in \mathscr{H}$ and $y' \in L^2(\mu \mid_{G_r \cup \Gamma})$ and a universal constant $C(< 2 \cdot 21^2)$ such that*

$$L = x \otimes y' \tag{2.59}$$

and $\max(\| x \|_2 , \| y' \|_2) \leqslant C \| L \|^{1/2}$.

Proof. Let $L \in P^\infty(\mu)_*$ with $\| L \| = 21^{-4}$. It suffices to show there exist x and y' (in the appropriate spaces) satisfying (2.59) with $\max(\| x \|_2 , \| y' \|_2) \leqslant 2$. Choose δ with $16 < \delta^{-1} < 21$. Let $a = b = 0$. By Lemma 6 there exist vectors $x_1 \in \mathscr{H}$ and $y_1 \in L^2(\mu \mid_{G_r \cup \Gamma})$ with $\| x_1 \|_2 < 17\delta$, $\| y_1 \| < 8\delta^2$, and $\| L - x_1 \otimes y_1 \| < \delta^8$.

By induction we now define vectors $x_2 , x_3 ,\ldots,$ in $\mathscr{H}$ and $y_2 , y_3 ,\ldots,$ in $L^2(\mu|_{G_r \cup \Gamma})$ satisfying

$$\| x_n \| < 17\delta^n,$$

$$\left\| \sum_{k=1}^{n} y_k \right\|_2 < \frac{1}{1 - 8\delta^n} \left\| \sum_{k=1}^{n-1} y_k \right\|_2 + 8\delta^{2n},$$

and

$$\left\| L - \left(\sum_{k=1}^{n} x_k \right) \otimes \left(\sum_{k=1}^{n} y_k \right) \right\|_* < \delta^{4(n+1)}.$$

A simple computation shows that $x = \sum_1^\infty x_k$ is in $\mathscr{H}$, $\| x \|_2 \leqslant 2$, and $\| \sum_1^n y_k \|_2 < 2$ for each n. Let $\{\sum_1^{n_p} y_k\}$ be a subsequence of $\{\sum_1^n y_k\}$ that converges weakly in $L^2(\mu \mid_{G_r \cup \Gamma})$, say to the vector y'.

Then for any function $\varphi \in P^\infty(\mu)$, we have

$$L(\varphi) = \lim \left\langle \varphi \sum_1^{n_p} x_k \,, \sum_1^{n_p} y_k \right\rangle$$

$$= \lim \left[\left\langle \varphi x, \sum_1^{n_p} y_k \right\rangle + \left\langle \varphi \left(\sum_1^{n_p} x_k - x \right), \sum_1^{n_p} y_k \right\rangle \right]$$

$$= \langle \varphi x, y' \rangle,$$

that is, $L = x \otimes y'$.

Remark. Let P denote the projection of $L^2(\mu)$ onto $\mathscr{H}$ and set $y = Py'$ where x and y' are given by Lemma 7. Using the natural map from $\mathfrak{A}(S)$ onto $P^\infty(\mu)$, we see that if $L \in (\mathfrak{A}(S), \sigma\text{-wot})^*$ there exist vectors x and y in $\mathscr{H}$ and a universal constant C such that

$$L(T) = \langle Tx, y \rangle$$

for all $T \in \mathfrak{A}(S)$ and $\max(\| x \|_2 \,, \| y \|_2) \leqslant C \| L \|^{1/2}$.

We are now ready to prove the main theorem of this section.

THEOREM 1. *Let S be a subnormal operator on $\mathscr{H}$ with minimal normal extension N on $\mathscr{K}$. If $L \in (\mathfrak{A}(S), \sigma\text{-wot})^*$, there exist vectors x and y in $\mathscr{H}$ and a universal constant C such that*

$$L(T) = \langle Tx, y \rangle \tag{2.60}$$

for all $T \in \mathfrak{A}(S)$ and

$$\max(\| x \|_2 \,, \| y_2 \|) \leqslant C \| L \|^{1/2}. \tag{2.61}$$

Proof. We first consider the case where S has a cyclic vector. Using a result of Bram [3], we can assume there exists a measure μ such that $N = M_z$ on $L^2(\mu)$. By Theorem 4.11 in [6] there exist a countable collection of mutually singular measures $\mu_0 \,, \mu_1 \,,\ldots$ such that $\mu = \sum_0 \mu_i$, and a countable collection of disjoint simply connected regions $G_1 \,, G_2 \,,\ldots,$ such that

$$P^\infty(\mu) = L^\infty(\mu_0) \oplus \bigoplus_{i=1} H^\infty(G_i \,, \mu_i).$$

By Theorem 7.1 in [6] there exist closed reducing subspaces $\mathscr{H}_0 \,, \mathscr{H}_1 \,,\ldots$ for S such that

$$S \mid_{\mathscr{H}_0} = M_z \quad \text{on} \quad L^2(\mu_0), \tag{2.62}$$

$$M_z \text{ on } L^2(\mu_i) \text{ is the minimal normal extension of } S \mid_{\mathscr{H}_i} \text{ for } i = 1, 2,\ldots, \tag{2.63}$$

$$P^\infty(\mu_i) = H^\infty(G_i) \qquad \text{for all} \quad i \geqslant 1. \tag{2.64}$$

[The reader should consult [6] for an explanation of the terminology used above.]

Fix $i \geqslant 1$ for this paragraph and let ψ be the conformal map of D onto G_i. Using [14, Lemma 4.4], we see that ψ can be extended to a map from $\bar{D}$ onto $\bar{G}_i$ in such a way so that if $\beta = \mu_i \circ \psi^{-1}$ and $\mathscr{L}_i = \{g \in L^2(\beta): g = f \circ \psi$ for $f \in \mathscr{H}_i\}$, then M_z on $L^2(\mu_i)$ is unitarily equivalent to M_ψ, multiplication by ψ, on $L^2(\beta)$; $S|_{\mathscr{H}_i}$ is unitarily equivalent to $M_\psi|_{\mathscr{L}_i}$ and $P^\infty(\beta) = H^\infty$. Using [10, Lemma 1], we see that ψ is a weak-star generator of H^∞. Hence $\mathfrak{A}(M_\psi|_{\mathscr{L}_i}) = H^\infty$ (where the equal sign, of course, means isometrically isomorphic and weak-star homeomorphic under the natural map). Clearly then $\mathscr{L}_i$ is also invariant under M_z (on $L^2(\beta)$). We also note that M_z on $L^2(\beta)$ is the minimal normal extension of $M_z|_{\mathscr{L}_i}$ by [6, Theorem 6.1]. It now follows, by the remark after Lemma 7, that there exist vectors x_i and y_i in $\mathscr{H}_i$ and a universal constant C such that if $L \in (\mathfrak{A}(S|_{\mathscr{H}_i}), \sigma\text{-wot})^*$, then

$$L(T) = \langle Tx_i, y_i \rangle \tag{2.65}$$

for all $T \in \mathfrak{A}(S|_{\mathscr{H}_i})$, and

$$\max(\|x_i\|, \|y_i\|) \leqslant C\|L\|^{1/2}. \tag{2.66}$$

Returning to the case at hand, we let $L \in (\mathfrak{A}(S), \sigma\text{-wot})^*$ and $L_i = L|_{\mathfrak{A}(S|_{\mathscr{H}_i})}$ for $i = 0, 1, \ldots$. Since $S|_{\mathscr{H}_0}$ is normal, clearly there exist vectors x_0 and y_0 in $\mathscr{H}_0$ with $\max(\|x_0\|, \|y_0\|) \leqslant C\|L_0\|^{1/2}$ such that $L_0(T) = \langle Tx_0, y_0 \rangle$ for all $T \in \mathfrak{A}(S|_{\mathscr{H}_0})$. Choose x_i and y_i in $\mathscr{H}_i$ satisfying (2.65) and (2.66) for L_i for all $i \geqslant 1$. We compute

$$\begin{aligned} \left\| \sum_{N_1}^{N_2} x_i \right\|_2^2 &= \sum_{N_1}^{N_2} \|x_i\|_2^2 \\ &\leqslant C^2 \sum_{N_1}^{N_1} \|L_i\|. \end{aligned} \tag{2.67}$$

Hence the parital sums of the infinite series $\sum_0 x_i$ are Cauchy. Let x denote its sum and using (2.67) again, we see $\|x\|_2 \leqslant C\|L\|^{1/2}$. Similar reasoning shows $y = \sum_0 y_i$ belongs to $\mathscr{H}$ and $\|y\|_2 \leqslant C\|L\|^{1/2}$. Clearly we also have

$$L(T) = \langle Tx, y \rangle$$

for all $T \in \mathfrak{A}(S)$.

Now let S be any subnormal on $\mathscr{H}$ with N its minimal normal extension on $\mathscr{K}$. Without loss of generality we can assume there exists a countable collection of measures $\{\mu_i\}$ such that $\mu_{i+1} \ll \mu_i$ and

$$N = \bigoplus M_{z_i} \qquad \text{on} \quad \bigoplus L^2(\mu_i).$$

Note that μ_1 is the scalar spectral measure for N. By the proof in [5, Proposition 4, Chap. 1], there exists $h = \sum \oplus h_i$ in $\mathscr{H}$ such that for each $g = \sum \oplus g_i$ in $\mathscr{H}$ we have

$$| g_i | \, d\mu_i \ll | h_i | \, d\mu_i$$

for each i. Note that μ_i and $| h_i | \, d\mu_i$ are mutually absolutely continuous because N is the minimal normal extension of S. Therefore h is a separating vector for $\mathscr{W}^*(N)$, so we can assume by changing basis that $\mathscr{H} \supset H^2(\mu_1)$, where $H^2(\mu_1)$ is the closure of the (analytic) polynomials in $L^2(\mu_1)$. (Consult [7, Chap. 4].) A simple application of the Stone–Weierstrass theorem shows M_z on $L^2(\mu_1)$ is the minimal normal extension of $S\,|_{H^2(\mu_1)}$. By the result of the preceding case we see that if $L \in (\mathscr{A}(S), \sigma\text{-wot})^*$ and $L' = L\,|_{\mathscr{A}(S|_{H^2(\mu_1)})}$, there exist vectors x and y in $H^2(\mu_1)$ such that

$$L'(M_\psi) = \langle M_\psi x, y \rangle$$

for all $\psi \in P^\infty(\mu_1)$ and $\max(\| x \|_2 , \| y \|_2) \leqslant C \| L' \|^{1/2}$, where C is the previous universal constant. By the properties of the scalar spectral measure μ_1, we then have

$$L(T) = \langle Tx, y \rangle$$

for all $T \in \mathscr{A}(S)$ and $\max(\| x \|_2 , \| y \|_2) \leqslant C \| L \|^{1/2}$.

3. Reflexivity of Subnormal Operators

Before we start we present a lemma that will be useful to future arguments. The proof is elementary.

Lemma 8. *Let S be a subnormal operator acting on a space $\mathscr{H}$ with M_z on $L^2(\mu)$ its minimal normal extension. Fix a vector $f \in \mathscr{H}$ and let $\mathscr{H}_f = \{ pf : p$ is a polynomial$\}^{-L^2(\mu)}$ and $\Delta = \{x \in \operatorname{spt} \mu : | f(x)| > 0\}$. Then the minimal normal extension of $S\,|_{\mathscr{H}_f}$ is M_z on $\chi_\Delta L^2(\mu)$.*

Proof. Let $\mathscr{K} = \{gf : g \in C(\operatorname{spt} \mu)\}^{-L^2(\mu)}$. By the Stone–Weierstrass theorem, M_z on $\mathscr{K}$ is the minimal normal extension of $S\,|_{\mathscr{H}_f}$. It is well known that every reducing subspace for M_z on $L^2(\mu)$ is of the form $\chi_G L^2(\mu)$ for some Borel set G. Let Δ' be a Borel set such that $\chi_{\Delta'} L^2(\mu) = \mathscr{K}$. Clearly $\Delta \supset \Delta'$.

Let $\Delta_n = \{x \in \operatorname{spt} \mu : | f(x)| \geqslant 1/n\}$. Using Lusin and Riesz's theorems, we can choose a sequence $\{ g_p \}$ of continuous functions such that $\| g_p \|_\infty \leqslant n$ and $g_p \to (1/f)\, \chi_{\Delta_n}(\mu)$ almost everywhere. By the Lebesgue-dominated convergence theorem we see that $\chi_{\Delta_n} \in \mathscr{K}$ for all n. Therefore $\Delta' = \Delta$.

We begin, as in Section 2 by considering a subnormal operator S on $\mathscr{H}$ whose minimal normal extension N is equal to M_z on $L^2(\mu)$. We also assume, to begin with, that $P^\infty(\mu) = H^\infty(D)$. As explained before, this allows us to normalize μ so that there exists a Borel set $\Gamma \subset \partial D$ such that

$$m\,|_\Gamma = \mu\,|_{\partial D}\,,$$

where m denotes normalized Lebesgue measure on ∂D.

Let $\{\Gamma_j\}$ be an increasing sequence of compact subsets of ∂D such that

$$m\left(\Gamma \backslash \left(\bigcup \Gamma_j\right)\right) = 0 = m\left(\left(\bigcup \Gamma_j\right) \backslash \Gamma\right).$$

We fix j. Let $\{J_n\}$ denote the countable collection of pairwise disjoint open subarcs of ∂D such that

$$\partial D \backslash \Gamma_j = \bigcup J_n\,.$$

Adding a finite number of points to Γ_j, if necessary, we may assume that the distance from zero to J_n is greater than $1 - 1/j$ for all n. (The reason for this technical assumption will be seen later. Let us also remark that we will be using the results of Section 2 with X corresponding to Γ_j.) Let I_n denote the chord joining the endpoints of J_n. Finally, let F_j denote the union of the regions bounded by $J_n \cup I_n$ (union over all n).

Only part of Lemma 6 was used to prove Theorem 1. The full extent of this result is the foundation for the proof of the next lemma.

LEMMA 9 (Notation as above). *Let $\{L_j\}$ be a sequence of elements in $P^\infty(\mu)_*$. Let $t \in L^2(\mu)$. There exist a sequence $\{b_j\}$ with $b_j \in L^2(\mu\,|_{F_j \cup \Gamma})$ for each j and a vector $\alpha \in \mathscr{H}$ so that*

$$L_j = (\alpha + t) \otimes b_j$$

for each j.

Proof. Without loss of generality we may assume that $\|L_j\| = \delta_j{}^4$ for each j, $\sum_j \delta_j$ is finite, $\pi_j (1 - 8\delta_j)^{-1}$ is finite, and $\delta_j < \frac{1}{16}$ for all j. Choose a doubly indexed sequence of numbers $\{\epsilon_{jk}\}$ so that $\epsilon_{jk} < \frac{1}{16}$ for each (j, k) and so small so that $\prod_{j,k} (1 - 8\epsilon_{jk})^{-1}$ and $\sum_{j,k} \epsilon_{jk}$ are finite. Let $B\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection such that $B(j, k) \leqslant B(j', k')$ if $j \leqslant j'$ and $k \leqslant k'$. (We visualize $\mathbb{N} \times \mathbb{N}$ as a matrix. Associate to the first column of the matrix, say at entry $(j, 1)$, the two numbers δ_j and ϵ_{j1}. At any other entry, say at (j, k), associate the number ϵ_{jk}. The main idea of the proof is to approximate L_j along the jth row of the matrix. If we are at position (j, k), we want to approximate L_j by $(t + \alpha_n) \otimes b_{jk}$ where $n = B(j, k)$, $\alpha_n \in \mathscr{H}$, and $b_{jk} \in L^2(\mu\,|_{F_j \cup \Gamma})$ so that $\{\alpha_n\}$ is a Cauchy sequence in $\mathscr{H}$, $\|L_j - (t + \alpha_n) \otimes b_{jk}\|_* \to 0$ as $k \to \infty$, and so that the sequence $\{\|b_{jk}\|\}$ is bounded.)

We proceed by induction (on the range of B). For $n = 1 = B(1, 1)$ we apply Lemma 6 with $a = t$, $b = 0$, $\delta = \delta_1$, $\epsilon = (\epsilon_{11})^4$ to find $a_{11} \in \mathscr{H}$, $b_{11} \in L^2(\mu\,|_{F_1 \cup \Gamma})$

so that $\| a_{11} \|_2 < 17\delta_1$, $\| b_{11} \| < 8\delta_1^2$, $| a_{11} + t | \geqslant (1 - 8\delta_1) | t |$ almost everywhere on Γ and

$$\| L_1 - (a_{11} + t) \otimes b_{11} \|_* < (\epsilon_{11})^4.$$

Let $\alpha_1 = a_{11}$.

The induction assumption goes as follows: if $n = B(j, 1)$ we assume that $a_{j1} \in \mathscr{H}$ and $b_{j1} \in L^2(\mu |_{F_j \cup \Gamma})$ have been found, $\alpha_n = \alpha_{n-1} + a_{j1}$, and

$$\| a_{j1} \|_2 < 17\delta_j ,$$

$$\| b_{j1} \|_2 < 8\delta_j^2,$$

$$| \alpha_n + t | \geqslant (1 - 8\delta_j)(| \alpha_{n-1} + t |)$$

almost everywhere on Γ,

$$\| L_j - (\alpha_n + t) \otimes b_{j1} \|_* < (\epsilon_{j1})^4,$$

and

$$\| L_{j'} - [(\alpha_n + t) \otimes b_{j'k'} \chi_D + (\alpha_{B(j',k')} + t) \otimes b_{j'k'} \chi_\Gamma] \|_* < (\epsilon_{j'k'})^4$$

for alll (j', k') with $B(j', k') \leqslant n$.

In the other possible case where $n = B(j, k)$ with $k > 1$, we assume that $a_{jk} \in \mathscr{H}$ and $b_{jk} \in L^2(\mu |_{F_j \cup \Gamma})$ have been obtained, $\alpha_n = \alpha_{n-1} + a_{jk}$,

$$\| a_{jk} \|_2 < 17\epsilon_{j,k-1} ,$$

$$\begin{aligned} \| b_{j,k} \| < {} & \left(\pi \frac{1}{1 - 8\delta_r}\right)\left(\pi \frac{1}{1 - 8\epsilon_{r,s-1}}\right) \| b_{j1} \|_2 \\ & + \left(\pi \frac{1}{1 - 8\delta_r}\right)\left(\pi \frac{1}{1 - 8\epsilon_{r,s-1}}\right)\left(\sum_{l=1}^{k-2} 8(\epsilon_{jl})^2\right) \\ & + 8(\epsilon_{j,k-1})^2, \end{aligned}$$

where the terms δ_r and ϵ_{rs} in these products are precisely those with $B(r, 1) < n$ and $B(r, s) < n$, respectively,

$$| \alpha_n + t | \geqslant (1 - 8\epsilon_{j,k-1}) | \alpha_{n-1} + t |$$

almost everywhere on Γ,

$$\| L_j - (\alpha_n + t) \otimes b_{jk} \|_* < (\epsilon_{jk})^4,$$

and

$$\| L_{j'} - [(\alpha_n + t) \otimes b_{j'k'} \chi_D + (\alpha_{B(j',k')} + t) \otimes b_{j'k'} \chi_\Gamma] \|_* \leqslant (\epsilon_{j'k'})^4$$

for all (j', k') with $B(j', k') \leqslant n$.

We now proceed with the inductive step. Case 1: suppose $n+1 = B(m, 1)$ for some $m > 1$. We apply Lemma 6 with $a = \alpha_n + t$, $b = 0$, $\delta = \delta_m$, $\epsilon = (\epsilon_{m1})^4$, $L = L_m$, ϵ' sufficiently small (to be determined later), and $\{d_1, \ldots, d_i\} = \{b_{jk}\}_{B(j,k)\leqslant n}$ to obtain $a_{m1} \in \mathscr{H}$, $b_{m1} \in L^2(\mu|_{F_m \cup \Gamma})$ so that

$$\| a_{m1} \|_2 < 17\delta_m,$$

$$\| b_{m1} \|_2 < 8\delta_m^2,$$

$$| \alpha_n + a_{m1} + t | \geqslant (1 - 8\delta_m) | \alpha_n + t |$$

almost everywhere on Γ,

$$\| L_m - (\alpha_n + a_{m1} + t) \otimes b_{m1} \|_* < (\epsilon_{m1})^4,$$

and (assuming the induction hypotheses) we choose ϵ' small enough so that

$$\| L_j - [(\alpha_n + a_{m1} + t) \otimes b_{jk}\chi_D + (\alpha_{B(j,k)} + t) \otimes b_{jk}\chi_\Gamma] \|_* < (\epsilon_{jk})^4$$

for all (j, k) with $B(j, k) \leqslant n$. Define $\alpha_{n+1} = \alpha_n + a_{m1}$.

Case 2: suppose $n + 1 = B(m, p)$ with $p > 1$. Note by the induction hypotheses we have

$$\| L_m - [((\alpha_n + t)\chi_D + (\alpha_{B(m,p-1)} + t)\chi_\Gamma) \otimes b_{m,p-1}] \|_* < (\epsilon_{m,p-1})^4.$$

Therefore if $a = \alpha_n + t$ and $b = b_{m,p-1}\chi_D + b_{m,p-1}((\bar{\alpha}_{B(m,p-1)} + \bar{t})/(\bar{\alpha}_n + \bar{t}))\chi_\Gamma$ (see the statement after inequality (3.1)), we have

$$\| L_m - a \otimes b \| < (\epsilon_{m,p-1})^4.$$

We apply Lemma 6 with $L = L_m$, a and b as defined above, $\delta = \epsilon_{m,p-1}$, $\epsilon = (\epsilon_{m,p})^4$ to obtain $a_{mp} \in \mathscr{H}$, $b_{mp} \in L^2(\mu|_{F_m \cup \Gamma})$ so that

$$\| a_{mp} \|_2 < 17\epsilon_{mp-1},$$

$$\| b_{mp} \|_2 < \frac{1}{1 - 8\epsilon_{m,p-1}} \| b \|_2 + 8(\epsilon_{m,p-1})^2,$$

$$| \alpha_n + a_{mp} + t | \geqslant (1 - 8\epsilon_{m,p-1}) | \alpha_n + t |$$

almost everywhere on Γ, and

$$\| L_m - (\alpha_n + a_{mp} + t) \otimes b_{mp} \|_* < (\epsilon_{mp})^4;$$

and we choose ϵ' of Lemma 6 so small (where the size of ϵ' is determined by the induction hypotheses) so that

$$\| L_j - [(\alpha_n + a_{mp} + t) \otimes b_{jk}\chi_D + (\alpha_{B(j,k)} + t) \otimes b_{jk}\chi_\Gamma] \|_* < (\epsilon_{jk})^4$$

for all (j, k) with $B(j, k) \leqslant n$. We set $\alpha_{n+1} = \alpha_n + a_{mp}$. This case will be finished if we obtain the desired estimate on $\| b_{mp} \|_2$.

Let us first observe that by induction hypotheses we have

$$| \alpha_n + t | \geqslant [\pi(1 - 8\delta_j)][\pi(1 - 8\epsilon_{j,k-1})] \, | \alpha_{B(m,p-1)} + t | \tag{3.1}$$

almost everywhere on Γ. The terms δ_j and ϵ_{jk} appearing in these products are precisely those with $B(m, p - 1) \leqslant B(j, 1) < B(m, p)$ and $B(m, p - 1) \leqslant B(j, k) < B(m, p)$, respectively. [This computation also shows why the term $b_{mp-1}((\bar{\alpha}_{B(m,p-1)} + \bar{t})/(\bar{\alpha}_n + \bar{t})) \chi_\Gamma$ in the definition of b belongs to $L^2(\mu)$.] Hence, by induction we have

$$\begin{aligned} \| b_{mp} \|_2 &< \frac{1}{1 - 8\epsilon_{m,p-1}} \left\| b_{m,p-1}\chi_D + b_{m,p-1} \left(\frac{\bar{\alpha}_{B(m,p-1)} + \bar{t}}{\bar{\alpha}_n + \bar{t}} \right) \chi_\Gamma \right\|_2 \\ &\quad + 8(\epsilon_{m,p-1})^2 \\ &\leqslant \frac{1}{1 - 8\epsilon_{m,p-1}} \left\| \left(\frac{\bar{\alpha}_{B(m,p-1)} + \bar{t}}{\bar{\alpha}_n + \bar{t}} \right) \chi_\Gamma \right\|_\infty \| b_{m,p-1} \|_2 \\ &\quad + 8(\epsilon_{m,p-1})^2 \\ &\leqslant \left(\pi \frac{1}{1 - 8\delta_j} \right) \left(\pi \frac{1}{1 - 8\epsilon_{j,k-1}} \right) \| b_{m1} \|_2 \\ &\quad + \left(\pi \frac{1}{1 - 8\delta_j} \right) \left(\pi \frac{1}{1 - 8\epsilon_{j,k-1}} \right) \left(\sum_{l=1}^{p-2} 8(\epsilon_{ml})^2 \right) \\ &\quad + 8(\epsilon_{m,p-1})^2 \end{aligned}$$

by using inequality (3.1). Also note the products involving δ_j's and $\epsilon_{j,k}$'s are over those j with $B(j, 1) < B(m, p)$ and those (j, k) with $B(j, k) < B(m, p)$.

It now follows that $\{\alpha_n\}$ is a Cauchy sequence in $\mathscr{H}$ because $\sum_j \delta_j$ and $\sum_{j,k} \epsilon_{jk}$ are finite. We let $\alpha = \lim \alpha_n$. It also follows that the $b_{j,k}$'s are uniformly bounded in $L^2(\mu)$ norm because all the appropriate infinite products are finite. By induction we now have

$$\lim_{p \to \infty} \| L_j - (\alpha_n + t) \otimes b_{jp} \| = 0.$$

If we let b_j be any weak cluster point of the sequence $\{b_{jp}\}_p$ then, by construction, $b_j \in L^2(\mu \,|_{F_j \cup \Gamma})$ and if we use an argument identical to that given at the end of the proof of Lemma 7, we have

$$L_j = (\alpha + t) \otimes b_j \,,$$

for each j. This finishes the proof.

We now recall a topological concept introduced in [11]. Let K be a compact subset of $\bar{D}$. Let U be a component of $D \backslash K$. Then $a \in \partial U \cap \partial D$ is called a strong boundary point of U if for each $\alpha \in (0, \pi)$, there exists an isosceles triangle $T_{a\alpha}$ such that a is a vertex of $T_{a\alpha}$; int $T_{a\alpha} \subset U$; the interior angle of $T_{a\alpha}$ at a has measure α; the radial line segment from zero to the point a bisects the interior angle at a. (The set of strong boundary points of a component U of $D\backslash K$ is a Borel set (Lemma 4, [11]).)

The next lemma follows from the construction of the set F_j.

LEMMA 10. *Every point of Γ_j that is not an endpoint of a complementary line segment is a strong boundary point of $D\backslash \bar{F}_j$.*

If τ is a measure, then $H^2(\tau)$ denotes the closure of the polynomials in $L^2(\tau)$. If $\lambda \in C$, then λ is called a point evaluation for $H^2(\tau)$ if there exists a constant C such that $|p(\lambda)| \leqslant C \| p \|_2$ for all polynomials p. In this case the smallest such C is called the norm of the point evaluation.

If λ is a point evaluation the map $p \to p(\lambda)$, p a polynomial, extends to a continuous linear functional on $H^2(\tau)$. For a function f in $H^2(\tau)$ the value of f under this linear functional will be denoted $\hat{f}(\lambda)$. The facts contained in the next lemma are well known.

LEMMA 11. *Let U be a bounded component of the complement of the support of a measure τ. Suppose one point in U is a bounded point evaluation for $H^2(\tau)$. Then every point in U is a bounded point evaluation for $H^2(\tau)$. The function $\hat{f}$ is analytic in U for every $f \in H^2(\tau)$. If $g \in H^2(\tau) \cap L^\infty(\tau)$, then $\hat{g}$ is a bounded analytic function in U; in fact,*

$$\sup_{z \in U} |\hat{g}(z)| \leqslant \| g \|_\tau .$$

If ν is a measure with ν carried by U and $g \in H^2(\tau + \nu)$, then

$$g|_U = \hat{g} \qquad \nu \text{ a.e.}$$

(Note: Since $g \in H^2(\tau + \nu)$ we have $g \in H^2(\tau)$. The function $\hat{g}$ in the last equality is computed when g is viewed as an element of $H^2(\tau)$.)

Proof. The proof of Lemma 4 in [10] shows that if one point evaluation exists in U, then every point in U gives rise to a point evaluation. Furthermore this proof shows that the norms of the point evaluations are uniformly bounded on compact subsets of U; hence, $\hat{f}$ is analytic on U for every $f \in H^2(\tau)$.

Suppose there exists a function $f \in H^2(\tau) \cap L^\infty(\tau)$ with $\| f \|_\tau < 1$ but there exists a $\lambda \in U$ such that $|\hat{f}(\lambda)| \geqslant 1$. For every positive integer n let $g_n = f^n$. An easy argument shows that $\hat{g}_n(\lambda) = [\hat{f}(\lambda)]^n$. Clearly $\| g_n \|_2$ goes to zero as $n \to \infty$. Hence, by continuity,

$$|\hat{g}_n(\lambda)| \to 0 \quad \text{as} \quad n \to \infty.$$

This is absurd.

The last part of the lemma is clearly true for polynomials. If $g \in H^2(\tau + \nu)$, choose a sequence $\{p_n\}$ of polynomials that converge in norm to g and that converge pointwise to g almost everywhere $(\tau + \nu)$. Hence,

$$\begin{aligned} g\,|_U &= \lim_{n\to\infty} p_n\,|_U \\ &= \lim_{n\to\infty} \hat{p}_n\,|_U \\ &= \hat{g}\,|_U \end{aligned}$$

almost everywhere (ν).

The proof of the next lemma can be found in [11].

LEMMA 12. *Let μ be a measure supported in $\bar{D}$, $P^\infty(\mu) = H^\infty$ and U a component of $D\backslash\operatorname{spt}\mu$. Suppose every point of U is a bounded point evaluation for $H^2(\mu)$. Then for every $f \in H^2(\mu) \cap L^\infty(\mu)$ the nontangential limit of $\hat{f}$ exists and equals $f(a)$ at almost every strong boundary point a of U.*

We now have the machinery that is needed to establish the fact that subnormal operators are reflexive. Notice in the proofs of the next three lemmas and the theorem a concrete representation of those subspaces in Lat S that are pertinent to the problem is given.

LEMMA 13. *Let S be a subnormal operator on $\mathscr{H}$ with minimal normal extension M_z on $L^2(\mu)$. Assume further that $P^\infty(\mu) = H^\infty$ and S has a cyclic vector. Then* Alg Lat $S \cap \{S\}' = \mathscr{W}(S)$.

Proof. We assume the construction performed at the beginning of this section has been done. In passing, we should point out that the normalization of making $\mu\,|_{\partial D} = m\,|_\Gamma$ prevents us from also assuming $\mathscr{H} = H^2(\mu)$.

Let $T \in$ Alg Lat $S \cap \{S\}'$. A result of Yoshino [16] implies that T lifts to an operator commuting with N; hence, there exists a function $\theta \in L^2(\mu)$ such that $T = M_\theta$, multiplication by θ, on $\mathscr{H}$.

Let e_0 denote the weak-star continuous linear functional on H^∞ of evaluation at 0. Let $L_j = e_0$ for all $j = 1, 2,\ldots$ and use Lemma 8 to find a vector $\alpha \in \mathscr{H}$ and a sequence $\{b_j\}$ with $b_j \in L^2(\mu\,|_{F_j \cup \Gamma})$ so that

$$e_0 = \alpha \otimes b_j$$

for all j. Let $\mathscr{H}_\alpha = \{p\alpha\colon p \text{ a polynomial}\}^{-L^2(\mu)}$ be the cyclic invariant subspace (for S) generated by α. Clearly $\mathscr{H}_\alpha \subset \mathscr{H}$. Let β be the measure defined by $d\beta = |\alpha|^2\, d\mu$. The map $p \to p\alpha$ where p is a polynomial can be extended to a linear isometry, say U, of $H^2(\beta)$ onto $\mathscr{H}_\alpha$ that induces a unitary equivalence between the operators M_z on these spaces. Moreover, we also have, for any j,

$$|p(0)| = \left| \int p\alpha\bar{b}_j\, d\mu \right| \leqslant \|b_j\|_2 \left[\int_{F_j \cup \Gamma} |p\alpha|^2\, d\mu \right]^{1/2}$$

so that the space $H^2(|\alpha|^2 d\mu|_{F_j \cup \Gamma})$ has bounded point evaluations throughout the component $D \backslash \bar{F}_j$ by Lemma 11. Clearly then $H^2(\beta)$ has point evaluations throughout the open unit disc D because of the properties of the sets F_j.

Since $\mathscr{H}_\alpha \in \text{Alg Lat } S$, it follows that M_θ on $\mathscr{H}_\alpha$ belongs to Alg Lat $(S|_{\mathscr{H}_\alpha})$. Using the unitary operator U, we see that M_θ on $H^2(\beta)$ resides in Alg Lat (M_z on $H^2(\beta)$). By Lemma 11 and the construction of the sets F_j, it follows that θ is a bounded analytic function on D. By Lemmas 11 and 12 and the properties of the sets F_j we have $\theta \in P^\infty(\mu)$. Therefore Alg Lat $S \cap \{S\}' \subset \mathscr{W}(S)$. Since the reverse inclusion is true for any operator the lemma is established.

LEMMA 14. *Let S be a subnormal operator acting on a space $\mathscr{H}$ with M_z on $L^2(\mu)$ as its minimal normal extension. Assume further that $P^\infty(\mu) = H^\infty$ and S has a cyclic vector. Then* Alg Lat $S \subset \{S\}'$. *Therefore, by Lemma* 13, *S is a reflexive operator.*

Proof. We use the results and the notation in the proof of the previous lemma.

Suppose $T \in \text{Alg Lat } S$ and let $T_1 = T|_{\mathscr{H}_\alpha}$ and $T_2 = U^{-1}T_1U$. Clearly then $T_2 \in \text{Alg Lat } (M_z \text{ on } H^2(\beta))$. We want to show T_2 commutes with M_z. For $\zeta \in D$, let k_ζ denote the vector in $H^2(\beta)$ such that $p(\zeta) = \langle p, k_\zeta \rangle$ for every polynomial p. Clearly k_ζ spans $[(z - \zeta) H^2(\beta)]^\perp$ because 1 is a cyclic vector.

Since $T_2 \in \text{Alg Lat}(M_z \text{ on } H^2(\beta))$ it follows that T_2^* leaves $[(z - \zeta) H^2(\beta)]^\perp$ invariant for $\zeta \in D$. Therefore there exists $\varphi(\zeta)$ in $\mathbb{C}$ such that

$$T_2^* k_\zeta = \overline{\varphi(\zeta)}\, k_\zeta .$$

It follows that

$$\langle T_2 p, k_\zeta \rangle = \varphi(\zeta) p(\zeta)$$

for each polynomial p. Because the polynomials are dense in $H^2(\beta)$, we have for all $g \in H^2(\beta)$ that

$$\langle zg, k_\zeta \rangle = \zeta \langle g, k_\zeta \rangle.$$

Thus

$$\begin{aligned} \langle z T_2 p, k_\zeta \rangle &= \zeta \varphi(\zeta) p(\zeta) \\ &= \langle T_2 z p, k_\zeta \rangle \end{aligned}$$

for all polynomials p and all $\zeta \in D$.

Let g be a function in $H^2(\beta)$. Since $\hat{g}(\zeta) = \langle g, k_\zeta \rangle$ and the polynomials are dense in $H^2(\beta)$, it follows that

$$(M_z T_2 g)^\wedge(\zeta) = (T_2 M_z g)^\wedge(\zeta)$$

for every ζ in D. Let $f = M_z T_2 g - T_2 M_z g$. By Szegö's theorem there exists an outer function θ in H^∞ such that $\theta f|_{\partial D} \in L^\infty(\beta|_{\partial D})$. Since $(\theta f)^\wedge = \hat{\theta} \hat{f} = 0$ almost everywhere (β) in D, it follows that $\theta f \in H^2(\beta) \cap L^\infty(\beta)$. Using Lemma 12 we conclude that f is the zero function. Thus $T_2M_z = M_zT_2$. Using the unitary operator U, we see that $TS = ST$ on the space $\mathscr{H}_\alpha$.

Now let h be a vector in $\mathscr{H}$. By Lemma 9, there exist a vector y in $\mathscr{H}_\alpha$ and a sequence of vectors $\{b_j\}$ with b_j in $L^2(\mu\,|_{(F_j\cup\Gamma)\cap\Delta})$ such that

$$e_0 = (h + y) \otimes b_j$$

for every j. (Here $\Delta = \{x \in \operatorname{spt}\mu\colon |\alpha(x)| > 0\}$. Lemma 9 applies because $P^\infty(\mu\,|_\Delta) = H^\infty$. This last equality follows because $H^2(\beta)$ has point evaluations throughout D.) Repeating the argument in the first part of this proof with $\alpha' = h + y$, we have $TS(h + y) = ST(h + y)$. Since $y \in \mathscr{H}_\alpha$ and T commutes with S on $\mathscr{H}_\alpha$, it follows that $TSh = STh$. Since h was an arbitrary vector in $\mathscr{H}$, the proof is done.

LEMMA 15. *Let S be a subnormal operator on a space $\mathscr{H}$ with N on $\mathscr{K}$ as its minimal normal extension. Let μ be a scalar spectral measure for N and assume $P^\infty(\mu) = H^\infty$. Then S is a reflexive operator.*

Proof. We can assume $N = \oplus M_z$ on $\oplus L^2(\mu_i)$ where $\{\mu_i\}$ is a countable collection of measures such that $\mu_{i+1} \ll \mu_i$ and $\mu_1 = \mu$ is the scalar spectral measure. Using the proof of Chaumat [5, Proposition 4, Chap. 1], we can find a vector $q' = \sum q_i$ in $\mathscr{H}$ such that $|q_i|\,d\mu$ and μ_i are mutually absolutely continuous. It now follows that there exist scalars λ_i such that $q \equiv \sum \lambda_i q_i$ belongs to $\mathscr{H}$ and $|q|\,d\mu_i$ and μ_i are mutually absolutely continuous.

Let $T \in \operatorname{Alg\,Lat} S$. Let $\mathscr{H}_q$ be the closure of the linear manifold $\{p(S)\,q\colon p$ a polynomial$\}$ in $\mathscr{H}$. Since $S\,|_{\mathscr{H}_q}$ has a cyclic vector and the scalar spectral measure for the minimal normal extension of this operator is μ, we can apply Lemma 14 to find a function $\theta_q \in P^\infty(\mu)$ such that $T\,|_{\mathscr{H}_q} = M_{\theta_q}$.

Suppose $t = \sum t_i$ is another vector in $\mathscr{H}$ with the same properties as q. Then $T\,|_{\mathscr{H}_t} = M_{\theta_t}$ for some $\theta_t \in P^\infty(\mu)$. It is easy to find a nonzero complex number λ such that $q_i + \lambda t_i$ is nonzero almost everywhere (μ_i) for all i. Let $r = q + \lambda t$. Then there exists θ_r in $P^\infty(\mu_1)$ so that $T\,|_{\mathscr{H}_r} = M_{\theta_r}$. Now

$$T(q + \lambda t) = \theta_q q + \lambda\theta_t t$$

and

$$T(r) = \theta_r q + \lambda\theta_r t.$$

Therefore

$$(\theta_r - \theta_q)q = \lambda(\theta_t - \theta_r)t.$$

Let $l = (\theta_r - \theta_q)q = \lambda(\theta_t - \theta_r)t$. Since $\theta_r - \theta_q$ and $\theta_t - \theta_r$ belong to $P^\infty(\mu)$, we have $l \in \mathscr{H}_q \cap \mathscr{H}_t$. On the set where l equals zero we have $\theta_r = \theta_t = \theta_q$. Now $Tl = \theta_q l = \theta_t l$ so that $\theta_t = \theta_q$ on the set where l is nonzero.

Now let $f = \sum f_i$ be any nonzero vector in $\mathscr{H}$. As before we can find *a* $\lambda \in \mathbb{C}$,

$\lambda \neq 0$ such that $f_i + \lambda q_i$ is nonzero almost everywhere (μ_i) for all i. Let $g = f + \lambda q$. By the argument above we have $Tg = \theta_q g$. But

$$\begin{aligned} T(g) &= Tf + \lambda Tq \\ &= Tf + \lambda \theta_q q. \end{aligned}$$

Therefore $Tf = \theta_q f$. Since $\theta_q \in P^\infty(\mu)$ and f was arbitrary, the proof is finished.

THEOREM 3. *Every subnormal operator is reflexive.*

Proof. Let S be a subnormal operator on $\mathscr{H}$ and N its minimal normal extension on $\mathscr{K}$. Let μ be the scalar spectral measure for N. By Theorem 4.11 in [6] there are countable collections of mutually singular measures, $\{\mu_i\}_{i\geqslant 0}$, and simply connected regions, $\{G_i\}_{i\geqslant 1}$, such that

$$P^\infty(\mu) = L^\infty(\mu_0) \oplus \sum_{i\geqslant 1} \oplus H^\infty(G_i, \mu_i).$$

By Theorem 7.1 in [6] there exist closed reducing subspaces $\mathscr{H}_i$, $i \geqslant 0$, for S, and closed reducing subspaces $\mathscr{K}_i$, $i \geqslant 1$, for N, such that

(a) $S|_{\mathscr{H}_0}$ is a reductive normal operator,

(b) $N|_{\mathscr{K}_i}$ is the minimal normal extension of $S|_{\mathscr{H}_i}$ for all $i \geqslant 1$,

(c) μ_i is the scalar spectral measure for N_i for all $i \geqslant 1$ (and $P^\infty(\mu_i) = H^\infty(G_i, \mu_i)$),

(d) $\mathscr{A}(S) = \mathscr{W}^*(S|_{\mathscr{H}_0}) \oplus \sum_{i\geqslant 1} \oplus \mathscr{A}(S|_{\mathscr{H}_i})$.

Hence by Theorem 2 we have

(e) $\mathscr{W}(S) = \mathscr{W}^*(S|_{\mathscr{H}_0}) \oplus \sum_{i\geqslant 1} \oplus \mathscr{W}(S|_{\mathscr{H}_i})$.

Since the spaces $\mathscr{H}_i$ reduce the operator S and $S|_{\mathscr{H}_0}$ is normal (therefore, reflexive [13]) it is sufficient to prove $S|_{\mathscr{H}_i}$ is reflexive for all $i \geqslant 1$.

Fix $i \geqslant 1$ and let φ be the conformal map of D onto G_i. Extend φ to the boundary of D by giving it its radial limits. By Lemma 4.4 in [14], Lemma 8.6 in [6], and Theorems 6.1 and 8.11 in [6], if ν is the measure defined on $\bar{D}$ by $\nu(E) = \mu(\varphi(E))$ for all Borel sets $E \subset \bar{D}$, then there exists a normal operator N_i with scalar spectral measure ν with $P^\infty(\nu) = H^\infty$, and an invariant subspace $\mathscr{L}_i$ for $\varphi(N_i)$ such that $S|_{\mathscr{H}_i}$ is unitarily equivalent to $\varphi(N_i)|_{\mathscr{L}_i}$, and $\varphi(N_i)$ is the minimal normal extension of this latter subnormal operator.

By Lemma 1 in [10] we know that φ is a weak-star generator of H^∞. Hence $\mathscr{L}_i$ is also invariant for N_i, $\mathscr{W}(\varphi(N_i)|_{\mathscr{L}_i}) = \mathscr{W}(N_i|_{\mathscr{L}_i})$, and furthermore N_i is the minimal normal extension of $N_i|_{\mathscr{L}_i}$ if we appeal to Theorem 6.1 in [6] again. Therefore, by Lemma 15, $N_i|_{\mathscr{L}_i}$ is reflexive. Since φ is a weak-star generator of H^∞, it follows that Lat $(N_i|_{\mathscr{L}_i}) = \mathrm{Lat}(\varphi(N_i)|_{\mathscr{L}_i})$. Hence $\varphi(N_i)|_{\mathscr{L}_i}$ is reflexive. The proof is done if we recall $S|_{\mathscr{H}_i}$ is unitarily equivalent to $\varphi(N_i)|_{\mathscr{L}_i}$.

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